

# Some generalization of Cauchy's and the quadratic functional equations

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**Abstract.** We find the solutions  $f, g, h: S \rightarrow H$  of each of the functional equations

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) = |\Lambda|g(x) + h(y), \quad x, y \in S,$$

where  $(S, +)$  is an abelian semigroup,  $\Lambda$  is a finite subgroup of the automorphism group of  $S$ ,  $(H, +)$  is an abelian group.

**Mathematics Subject Classification (2010).** 39B52.

**Keywords.** Cauchy functional equation, quadratic functional equation.

## 1. Introduction

The generalization of the quadratic functional equation

$$f(x + y) + f(x + \sigma y) = 2f(x) + 2f(y), \quad x, y \in G,$$

where  $\sigma$  is an automorphism of the abelian group  $G$  such that  $\sigma^2 = id_G$ ,  $f, g: G \rightarrow \mathbb{C}$  was investigated by Stetkær [3].

In Stetkær [4] solved the functional equation

$$\frac{1}{N} \sum_{n=0}^{N-1} f(z + \omega^n \zeta) = g(z) + h(\zeta), \quad z, \zeta \in \mathbb{C},$$

where  $N \in \{2, 3, \dots\}$  and  $\omega$  is a primitive  $N$ th root of unity,  $f, g, h: \mathbb{C} \rightarrow \mathbb{C}$  are continuous.

In the present paper we give the complete solution of the following functional equation

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) = Lg(x) + h(y), \quad x, y \in S,$$

where  $(S, +)$  is an abelian semigroup,  $\Lambda$  is a finite subgroup of the automorphism group of  $S$ ,  $L = \text{card}\Lambda$ ,  $(H, +)$  is an abelian group uniquely divisible by  $(L + 1)!$ ,  $f, g, h: S \rightarrow H$ .

## 2. Main result

In this work we use a theorem and a corollary proved by Mazur and Orlicz [2], and generalized by Djoković [1]:

**Theorem 1.** *Let  $(S, +)$  be an abelian semigroup,  $n \in \mathbb{N}$ ,  $(H, +)$  be an abelian group uniquely divisible by  $(n + 1)!$ ,  $f: S \rightarrow H$  satisfies the equation*

$$\Delta_v^{n+1} f(u) = 0, \quad u, v \in S.$$

*Then there exist  $A_0 \in H$  and  $k$ -additive, symmetric mappings  $A_k: S \rightarrow H$ ,  $k \in \{1, \dots, n\}$  such that*

$$f(x) = A_0 + A_1(x) + \dots + A_n(x, \dots, x), \quad x \in S.$$

**Corollary 1.** *Let  $(S, +)$  be an abelian semigroup,  $n \in \mathbb{N}$ ,  $(H, +)$  be an abelian group uniquely divisible by  $(n + 1)!$ ,  $f, g: S \rightarrow H$  satisfy the equation*

$$\Delta_v^n f(u) = g(v), \quad u, v \in S.$$

*Then there exist  $A_0 \in H$  and  $k$ -additive, symmetric mappings  $A_k: S \rightarrow H$ ,  $k \in \{1, \dots, n\}$  such that*

$$\begin{aligned} f(x) &= A_0 + A_1(x) + \dots + A_n(x, \dots, x), \quad x \in S, \\ g(x) &= n! A_n(x, \dots, x), \quad x \in S. \end{aligned}$$

We start with some lemmas.

**Lemma 1.** *For all  $n \in \mathbb{N}$  we have*

$$\sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^i = 0, \quad i \in \{1, \dots, n-1\}, \quad n \neq 1, \quad (1)$$

$$\sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^n = n!. \quad (2)$$

*Proof.* First we prove (1) by induction on  $n$ .

For  $n = 2$  we have

$$\sum_{k=1}^2 (-1)^{2-k} \binom{2}{k} k^i = -2 \cdot 1 + 1 \cdot 2 = 0.$$

Assume that (1) holds for  $2, \dots, n$ . Since

$$k \binom{n+1}{k} = (n+1) \binom{n}{k-1}, \quad k \in \{1, \dots, n\},$$

we have

$$\begin{aligned}\sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} k &= (n+1) \sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n}{k-1} \\ &= (n+1) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} = 0.\end{aligned}$$

Hence, for  $i \in \{1, \dots, n-1\}$ ,

$$\begin{aligned}\sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} k^{i+1} &= (n+1) \sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n}{k-1} k^i \\ &= (n+1) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (k+1)^i = (n+1) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \sum_{m=0}^i \binom{i}{m} k^m \\ &= (n+1) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} + (n+1) \sum_{m=1}^i \binom{i}{m} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^m \\ &= (n+1) \sum_{m=1}^i \binom{i}{m} \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^m = 0.\end{aligned}$$

Now we prove (2) by induction on  $n$ .

For  $n = 1$  we have

$$\sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^n = 1.$$

Assume that (2) holds for some  $n \in \mathbb{N}$ . Using (1) we get

$$\begin{aligned}\sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} k^{n+1} &= (n+1) \sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n}{k-1} k^n \\ &= (n+1) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (k+1)^n = (n+1) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \sum_{m=0}^n \binom{n}{m} k^m \\ &= (n+1) \sum_{m=0}^n \binom{n}{m} \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^m = (n+1) \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^n \\ &= (n+1)!.\end{aligned}$$

□

**Lemma 2.** *Let  $(H, +)$  be an abelian group uniquely divisible by  $n!$ ,  $x_1, \dots, x_n \in H$  be such that*

$$\sum_{i=1}^n k^i x_i = 0, \quad k \in \{1, \dots, n\}.$$

*Then  $x_1 = \dots = x_n = 0$ .*

*Proof.* First, we notice that for the group  $(H, +)$  from the unique divisibility by  $n!$  we obtain the unique divisibility by  $k!$  for each  $k \in \{1, \dots, n\}$ .

We prove this theorem by induction on  $n$ .

For  $n = 1$  we have  $x_1 = 0$ . Assume that the theorem holds for  $n - 1, n > 1$ . In view of Lemma 1 we have

$$0 = \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} \sum_{i=1}^n k^i x_i = \sum_{i=1}^n \left[ \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^i \right] x_i = n! x_n.$$

Hence  $x_n = 0$  and

$$\sum_{k=1}^{n-1} k^i x_i = 0.$$

By the induction hypothesis we obtain

$$x_1 = \dots = x_{n-1} = 0 = x_n,$$

which finishes the proof.  $\square$

**Lemma 3.** *Let  $(S, +)$  be an abelian semigroup,  $\Lambda$  be a finite subgroup of the automorphism group of  $S$ ,  $L := \text{card} \Lambda$ ,  $(H, +)$  be an abelian group uniquely divisible by  $L!$ . Further, let for each  $k \in \{1, \dots, L\}$   $A_k: S \rightarrow H$  be  $k$ -additive and symmetric mappings,  $f: S \rightarrow H$  be a function given by the formula*

$$f(x) = A_1(x) + \dots + A_L(x, \dots, x), \quad x \in S.$$

*Then a function  $f$  satisfies the equation*

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) = Lf(x) + \sum_{\lambda \in \Lambda} f(\lambda y), \quad x, y \in S \quad (3)$$

*if and only if*

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k-1, \quad 2 \leq k \leq L.$$

*Proof.* First we show that for  $x, y \in S$

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) - \sum_{\lambda \in \Lambda} f(\lambda y) - Lf(x) = \sum_{\lambda \in \Lambda} \sum_{i=1}^{L-1} \sum_{k=i+1}^L \binom{k}{i} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i).$$

Indeed, we have the following sequence of identities

$$\begin{aligned}
 & \sum_{\lambda \in \Lambda} f(x + \lambda y) - \sum_{\lambda \in \Lambda} f(\lambda y) - Lf(x) \\
 &= \sum_{\lambda \in \Lambda} \sum_{k=1}^L [A_k(x + \lambda y, \dots, x + \lambda y) - A_k(\lambda y, \dots, \lambda y) - A_k(x, \dots, x)] \\
 &= \sum_{\lambda \in \Lambda} \sum_{k=2}^L \sum_{i=1}^{k-1} \binom{k}{i} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) \\
 &= \sum_{\lambda \in \Lambda} \sum_{i=1}^{L-1} \sum_{k=i+1}^L \binom{k}{i} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i), \quad x, y \in S.
 \end{aligned}$$

By the above equality we obtain that if

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k-1, \quad 1 \leq k \leq L,$$

then  $f$  satisfies (3).

Now we assume that  $f$  satisfies (3). Then

$$\sum_{\lambda \in \Lambda} \sum_{i=1}^{L-1} \sum_{k=i+1}^L \binom{k}{i} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0.$$

For each  $i \in \{1, \dots, L-1\}$  we define  $g_i: S \times S \rightarrow H$  by the formula

$$g_i(x, y) := \sum_{\lambda \in \Lambda} \sum_{k=i+1}^L \binom{k}{i} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i), \quad x, y \in S.$$

Since

$$g_i(x, my) = m^i g_i(x, y), \quad m \in \mathbb{N}, \quad x, y \in S, \quad 1 \leq i \leq L-1,$$

then

$$\sum_{i=1}^{L-1} m^i g_i(x, y) = \sum_{i=1}^{L-1} g_i(x, my) = 0, \quad m \in \mathbb{N}, \quad x, y \in S.$$

In view of Lemma 2 we obtain

$$g_i(x, y) = 0, \quad x, y \in S, \quad i \in \{1, \dots, L-1\}.$$

Now, for each  $i \in \{1, \dots, L-1\}, j \in \{1, \dots, L-i\}$ , we define  $h_{i,j}: S \times S \rightarrow H$  by the formula

$$h_{i,j} := \binom{i+j}{i} \sum_{\lambda \in \Lambda} A_{i+j}(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i), \quad x, y \in S.$$

Since

$$h_{i,j}(mx, y) = m^j h_{i,j}(x, y), \quad m \in \mathbb{N}, \quad x, y \in S, \quad 1 \leq j \leq L - i, \quad 1 \leq i \leq L - 1,$$

we have

$$\sum_{j=1}^{L-i} m^j h_{i,j}(x, y) = \sum_{j=1}^{L-i} h_{i,j}(mx, y) = g_i(mx, y) = 0,$$

for  $m \in \mathbb{N}$ ,  $x, y \in S$ ,  $1 \leq i \leq L - 1$ . In view of lemma 2 we obtain

$$h_{i,j}(x, y) = 0, \quad x, y \in S, \quad 1 \leq j \leq L - i, \quad 1 \leq i \leq L - 1,$$

so we get

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k - 1, \quad 2 \leq k \leq L.$$

□

**Theorem 2.** Let  $(S, +)$  be an abelian semigroup,  $\Lambda$  be a finite subgroup of the automorphism group of  $S$ ,  $L := \text{card} \Lambda$ ,  $(H, +)$  be an abelian group. Further, let  $f: S \rightarrow H$  satisfy Eq. (3),  $g: S \rightarrow H$  be a function given by

$$g(x) := - \sum_{i=0}^{L-1} (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} f \left( \sum_{\mu \in \Lambda_{i,j}} \mu x \right), \quad x \in S,$$

where  $\Lambda_{i,j} \subset \Lambda$  are pairwise different sets such that  $\text{card} \Lambda_{i,j} = L - i$  for  $j \in \{1, \dots, \binom{L}{i}\}$ ,  $i \in \{0, \dots, L\}$ . Then

$$L \Delta_v^L f(u) = Lg(v), \quad u, v \in S.$$

*Proof.* First we observe that any solution of Eq. (3) in a semigroup without zero can be uniquely extended to a solution in a semigroup with a neutral element 0 by putting  $f(0) = 0$ . So we can assume that  $(S, +)$  is a monoid.

Since

$$\lambda \Lambda_{i,j} = \Lambda_{i,k} \Rightarrow \lambda^{-1} \Lambda_{i,k} = \Lambda_{i,j}, \quad \lambda \in \Lambda, i \in \{0, \dots, L\}, j, k \in \left\{1, \dots, \binom{L}{i}\right\},$$

we have

$$\sum_{j=1}^{\binom{L}{i}} \sum_{\lambda \in \Lambda} f \left( \sum_{\mu \in \Lambda_{i,j}} \lambda \mu x \right) = L \sum_{j=1}^{\binom{L}{i}} f \left( \sum_{\mu \in \Lambda_{i,j}} \mu x \right), \quad x \in S. \quad (4)$$

Fix  $u, v \in S$ . Let

$$x_i := u + iv, \quad y_{i,j} := \sum_{\mu \in \Lambda_{i,j}} \mu v, \quad j \in \left\{1, \dots, \binom{L}{i}\right\}, \quad i \in \{0, \dots, L\}.$$

For  $\lambda \in \Lambda, i \in \{0, \dots, L\}, j \in \{1, \dots, \binom{L}{i}\}$  we consider two cases:

- (i)  $\lambda^{-1} \in \Lambda_{i,j}$ . Hence  $i \neq L$ . Let  $k \in \{1, \dots, \binom{L}{i+1}\}$  be such that  $\Lambda_{i,j} = \Lambda_{i+1,k} \cup \{\lambda^{-1}\}$ . Then

$$\begin{aligned} x_i + \lambda y_{i,j} &= u + iv + \sum_{\mu \in \Lambda_{i,j}} \lambda \mu v = u + (i+1)v + \sum_{\mu \in \Lambda_{i+1,k}} \lambda \mu v \\ &= x_{i+1} + \lambda y_{i+1,k}. \end{aligned}$$

- (ii)  $\lambda^{-1} \notin \Lambda_{i,j}$ . Hence  $i \neq 0$ . Let  $k \in \{1, \dots, \binom{L}{i-1}\}$  be such that  $\Lambda_{i-1,k} = \Lambda_{i,j} \cup \{\lambda^{-1}\}$ . Then

$$\begin{aligned} x_i + \lambda y_{i,j} &= u + iv + \sum_{\mu \in \Lambda_{i,j}} \lambda \mu v = u + (i-1)v + \sum_{\mu \in \Lambda_{i-1,k}} \lambda \mu v \\ &= x_{i-1} + \lambda y_{i-1,k}. \end{aligned}$$

From the above consideration and equality (4) we obtain

$$\begin{aligned} 0 &= \sum_{i=0}^L (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} \sum_{\lambda \in \Lambda} f(x_i + \lambda y_{i,j}) \\ &= \sum_{i=0}^L (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} \left[ Lf(x_i) + \sum_{\lambda \in \Lambda} f(\lambda y_{i,j}) \right] \\ &= L \sum_{i=0}^L (-1)^{L-i} \binom{L}{i} f(u + iv) + \sum_{i=0}^{L-1} (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} \sum_{\lambda \in \Lambda} f \left( \sum_{\mu \in \Lambda_{i,j}} \lambda \mu v \right) \\ &= L \sum_{i=0}^L (-1)^{L-i} \binom{L}{i} f(u + iv) + L \sum_{i=0}^{L-1} (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} f \left( \sum_{\mu \in \Lambda_{i,j}} \lambda \mu v \right) \\ &= L \Delta_v^L f(u) - Lg(v). \end{aligned}$$

□

**Theorem 3.** Let  $(S, +)$  be an abelian semigroup,  $\Lambda$  be a finite subgroup of the automorphism group of  $S, L := \text{card} \Lambda, (H, +)$  be an abelian group uniquely divisible by  $(L+1)!$ . Then a function  $f: S \rightarrow H$  satisfies Eq. (3) if and only if there exist  $k$ -additive and symmetric mappings  $A_k: S^k \rightarrow H, k \in \{1, \dots, L\}$  such that

$$f(x) = A_1(x) + \cdots + A_L(x, \dots, x), \quad x \in S,$$

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k-1, \quad 2 \leq k \leq L.$$

*Proof.* The proof of this theorem follows straight from Theorem 2, Corollary 1 and Lemma 3.  $\square$

Now we show some applications of the above theorems.

**Theorem 4.** *Let  $(S, +)$  be an abelian semigroup,  $\Lambda$  be a finite subgroup of the automorphism group of  $S$ ,  $L := \text{card} \Lambda$ ,  $(H, +)$  be an abelian group uniquely divisible by  $(L+1)!$ . Then a function  $f: S \rightarrow H$  satisfies the equation*

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) = Lf(x) + Lf(y), \quad x, y \in S \quad (5)$$

*if and only if there exist  $k$ -additive, symmetric mappings  $A_k: S^k \rightarrow H$ ,  $k \in \{1, \dots, L\}$  such that*

$$f(x) = A_1(x) + \cdots + A_L(x, \dots, x), \quad x \in S,$$

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k-1, \quad 2 \leq k \leq L,$$

$$A_k(\mu x, \dots, \mu x) = A_k(x, \dots, x), \quad x \in S, \quad \mu \in \Lambda, \quad 1 \leq k \leq L.$$

*Proof.* It is easy to check that if  $f$  satisfies the three above conditions then  $f$  satisfies Eq. (5). We observe that if  $f$  satisfies Eq. (5), then  $f$  satisfies Eq. 3 and  $f(\mu x) = f(x)$ ,  $\mu \in \Lambda$ ,  $x \in S$ .

Indeed, for  $x, y \in S$ ,  $\mu \in \Lambda$  we have

$$Lf(y) = -Lf(x) + \sum_{\lambda \in \Lambda} f(x + \lambda y) = -Lf(x) + \sum_{\lambda \in \Lambda} f(x + \lambda \mu y) = Lf(\mu y),$$

which gives  $f(\mu x) = f(x)$ . Hence  $Lf(y) = \sum_{\lambda \in \Lambda} f(\lambda y)$  for  $y \in S$ , thus  $f$  satisfies Eq. (3). In view of Theorem 3 we obtain the first and the second condition of the thesis. Finally, we show that

$$A_k(\mu x, \dots, \mu x) = A_k(x, \dots, x), \quad x \in S, \quad \mu \in \Lambda, \quad 1 \leq k \leq L.$$

Fix  $\mu \in \Lambda$ . We observe that

$$0 = f(\mu x) - f(x) = \sum_{k=1}^L [A_k(\mu x, \dots, \mu x) - A_k(x, \dots, x)], \quad x \in S.$$

For each  $i \in \{1, \dots, L\}$  we define  $g_i: S \rightarrow H$  by the formula

$$g_i(x) := A_i(\mu x, \dots, \mu x) - A_i(x, \dots, x), \quad x \in S.$$

Since

$$g_i(mx) = m^i g_i(x), \quad m \in \mathbb{N}, \quad x \in S,$$



we have

$$0 = \sum_{i=1}^L g_i(mx) = \sum_{i=1}^L m^i g_i(x), \quad m \in \mathbb{N}, \quad x \in S.$$

Hence, in view of Lemma 2, we obtain

$$g_i(x) = 0, \quad 1 \leq i \leq L, \quad x \in S,$$

which ends the proof.  $\square$

**Theorem 5.** Let  $(S, +)$  be an abelian semigroup,  $\Lambda$  be a finite subgroup of the automorphism group of  $S$ ,  $L := \text{card} \Lambda$ ,  $(H, +)$  be an abelian group uniquely divisible by  $L!$ . Then a function  $f: S \rightarrow H$  satisfies the equation

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) = Lf(x), \quad x, y \in S \quad (6)$$

if and only if there exist  $k$ -additive, symmetric mappings  $A_k: S^k \rightarrow H$ ,  $k \in \{1, \dots, L-1\}$  and  $A_0 \in H$  such that

$$f(x) = A_0 + A_1(x) + \dots + A_{L-1}(x, \dots, x), \quad x \in S,$$

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k, \quad 2 \leq k \leq L-1.$$

*Proof.* Let us note that if a function  $f$  satisfies the above conditions then  $f$  satisfies Eq. (6).

First we observe that if  $(S, +)$  is a semigroup without a neutral element 0, then we can uniquely extend each solution of (6) to a solution on the semigroup  $(S \cup \{0\}, +)$ . Indeed, we have

$$\begin{aligned} L \sum_{\lambda \in \Lambda} f(\lambda y) &= \sum_{\lambda \in \Lambda} \sum_{\mu \in \Lambda} f(\lambda y + \mu x) \\ &= \sum_{\mu \in \Lambda} \sum_{\lambda \in \Lambda} f(\mu x + \lambda y) = L \sum_{\mu \in \Lambda} f(\mu x), \quad x, y \in S, \end{aligned}$$

and

$$Lf \left( \sum_{\lambda \in \Lambda} \lambda y \right) = \sum_{\mu \in \Lambda} f \left( \mu \sum_{\lambda \in \Lambda} \lambda y \right) = \sum_{\mu \in \Lambda} f(\mu x), \quad x, y \in S.$$

Putting  $f(0) := f(\sum_{\lambda \in \Lambda} \lambda y)$  for  $y \in S$ . It is easy to check that the solution extended in this way satisfies Eq. 3 for the semigroup  $S$  with zero.

So we can assume that  $(S, +)$  is a monoid. Since

$$Lf(0) = \sum_{\lambda \in \Lambda} f(\lambda y), \quad y \in S,$$

then  $f_0 := f - f(0)$  satisfies Eq. 3. Let  $\Lambda_{i,j} \subset \Lambda, j \in \{1, \dots, \binom{L}{i}\}, i \in \{0, \dots, L\}$ , be sets from Theorem 2. We observe that for  $x \in S$  we have

$$\sum_{\lambda \in \Lambda} f_0 \left( \sum_{\mu \in \Lambda_{i,j}} \lambda \mu x \right) = \sum_{\lambda \in \Lambda} f \left( \lambda \sum_{\mu \in \Lambda_{i,j}} \mu x \right) - Lf(0) = 0.$$

By equality (4) we obtain

$$L \sum_{i=0}^{L-1} (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} f_0 \left( \sum_{\mu \in \Lambda_{i,j}} \mu x \right) = \sum_{i=0}^{L-1} (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} \sum_{\lambda \in \Lambda} f_0 \left( \sum_{\mu \in \Lambda_{i,j}} \lambda \mu x \right) = 0.$$

Then Theorem 2 holds with  $g = 0$ . In view of Theorem 1, Lemma 3 and putting  $A_0 := f(0)$  we obtain that there exist  $k$ -additive and symmetric mappings  $A_k: S^k \rightarrow H, k \in \{1, \dots, L-1\}$  such that

$$f(x) = A_0 + A_1(x) + \dots + A_{L-1}(x, \dots, x), \quad x \in S,$$

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k-1, \quad 2 \leq k \leq L-1,$$

Finally, we show that

$$\sum_{\lambda \in \Lambda} A_k(\lambda x, \dots, \lambda x) = 0, \quad x \in S, \quad 2 \leq k \leq L-1.$$

For each  $i \in \{1, \dots, L-1\}$  we define  $g_i: S \rightarrow H$  by the formula

$$g_i(x) := \sum_{\lambda \in \Lambda} A_i(\lambda x, \dots, \lambda x), \quad x \in S.$$

Since

$$g_i(mx) = m^i g_i(x), \quad m \in \mathbb{N}, \quad x \in S, \quad 1 \leq i \leq L-1,$$

we have

$$0 = \sum_{\lambda \in \Lambda} f(\lambda x) - Lf(0) = \sum_{i=1}^L g_i(mx) = \sum_{i=1}^L m^i g_i(x), \quad m \in \mathbb{N}, \quad x \in S.$$

In view of Lemma 2

$$g_i(x) = 0, \quad 1 \leq i \leq L-1, \quad x \in S,$$

which ends the proof.  $\square$

Our main result reads as follows

**Theorem 6.** *Let  $(S, +)$  be an abelian monoid,  $\Lambda$  be a finite subgroup of the automorphism group of  $S, L := \text{card} \Lambda, (H, +)$  be an abelian group uniquely divisible by  $(L+1)!$ . Then functions  $f, g, h: S \rightarrow H$  satisfy the equation*

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) = Lg(x) + h(y), \quad x, y \in S \quad (7)$$

if and only if there exist  $k$ -additive, symmetric mappings  $A_k: S^k \rightarrow H$ ,  $k \in \{1, \dots, L\}$  and  $A_0, B_0 \in H$  such that

$$\begin{aligned} f(x) &= A_0 + A_1(x) + \dots + A_L(x, \dots, x), \quad x \in S, \\ g(x) &= B_0 + A_1(x) + \dots + A_L(x, \dots, x), \quad x \in S, \\ h(x) &= LA_0 - LB_0 + \sum_{\lambda \in \Lambda} A_1(\lambda x) + \dots + \sum_{\lambda \in \Lambda} A_L(\lambda x, \dots, \lambda x), \quad x \in S, \\ \binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) &= 0, \quad x, y \in S, \quad 1 \leq i \leq k-1, \quad 2 \leq k \leq L. \end{aligned}$$

*Proof.* It is easy to check that if the four equalities above hold, then functions  $f, g, h$  satisfy Eq. (7).

Assume that functions  $f, g, h$  satisfy Eq. (7). First we observe that

$$\begin{aligned} Lf(0) &= Lg(0) + h(0), \\ Lf(x) &= Lg(x) + h(0) = Lg(x) - Lg(0) + Lf(0), \quad x \in S. \end{aligned}$$

Therefore, we get

$$f(x) = g(x) - g(0) + f(0), \quad x \in S, \quad (8)$$

and

$$Lg(0) + h(y) = \sum_{\lambda \in \Lambda} f(\lambda y) = \sum_{\lambda \in \Lambda} [g(\lambda y) - g(0)] + Lf(0), \quad y \in S,$$

hence

$$h(y) = \sum_{\lambda \in \Lambda} [g(\lambda y) - g(0)] + Lf(0) - Lg(0), \quad y \in S. \quad (9)$$

Let  $g_0 := g - g(0)$ . From the above equalities we have

$$\begin{aligned} \sum_{\lambda \in \Lambda} g_0(x + \lambda y) &= \sum_{\lambda \in \Lambda} [g(x + \lambda y) - g(0) + f(0)] - Lf(0) \\ &= \sum_{\lambda \in \Lambda} f(x + \lambda y) - Lf(0) = Lg(x) + h(y) - Lf(0) \\ &= Lg_0(x) + \sum_{\lambda \in \Lambda} g_0(\lambda y), \quad x, y \in S. \end{aligned}$$

In view of Theorem 3, there exist  $k$ -additive and symmetric mappings  $A_k: S^k \rightarrow H, k \in \{1, \dots, L\}$  such that

$$\begin{aligned} g_0(x) &= A_1(x) + \dots + A_L(x, \dots, x), \quad x \in S, \\ \binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) &= 0, \quad x, y \in S, \quad 1 \leq i \leq k-1, \quad 2 \leq k \leq L. \end{aligned}$$

Let  $B_0 := g(0), A_0 := f(0)$ . Then from equalities 8 and 9 we obtain the thesis.  $\square$

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Received: November 12, 2010

Revised: November 9, 2011